Not choosing is still a choice

Constructive mathematics without any choice

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Abstract

The axiom of choice (AC) states that every total relation contains a function. It enjoys a pivotal role in both classical and constructive dialects of mathematics. In the former, it is seen as a useful closure property invoked especially in set-theoretic contexts, in the latter it is seen either as a tautology, following from a constructive reading of totality proofs, or as a taboo, as by an extensional reading of totality proofs it enforces full classical logic. It has therefore been debated how much of AC should be accepted in constructive foundations and authors like Richman argued for

"Constructive mathematics without choice"

where even countable choice, not immediately jeopardising constructive reasoning, is avoided.

With this paper, we propose a continuation of Richman's programme of more radical extent and systematically study constructive foundations absent of countable, unique, or quantifier-free choice principles as well as the spurious fragments of (the actual) AC in form of extensionality principles:

"Constructive mathematics without *any* choice"

We argue that such a minimalistic setting is advantageous, for instance for studies in constructive reverse mathematics and synthetic computability theory. Apart from these programmatic considerations and a careful encyclopedia of choice principles, we revisit and refine several results from the literature: We show that already the partition principle (a consequence of AC of unknown strength) implies the excluded middle, that already logically decidable (inductive) equality of propositions implies proof irrelevance, and that function inversion principles such as the Cantor-Bernstein theorem not only rely on the excluded middle but also on unique choice. To the best of our knowledge, the latter is the first reverse mathematics result regarding the full axiom of unique choice, enabled by our minimal setting. Implementing such a minimalistic foundation, the proofs of all our results have been mechanised with the Rocq prover.

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Supplementary Material *Formalisation (Source Code)*:
`https://github.com/inria-cambium/not-choosing-is-still-a-choice`

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1 Introduction

In classical mathematics, the axiom of choice (AC) has a special status: although its seemingly innocent formulations state that every total relation contains a function, that every two sets are comparable in cardinality, or that the cartesian product of non-empty sets is non-empty, it also has drastic consequences like the existence of non-measurable sets, the related Banach-Tarski paradox, or that every set can be well-ordered [34]. To establish the latter, Zermelo formulated AC as a key assumption in his axiomatic system of set theory [69] and delivered a second proof and justification for his use of AC after facing sincere criticism [70]. Since then, AC has been proved independent from Zermelo's other axioms in a line of work spanning from Gödel [27] to Cohen [10], and nowadays remains a subject of study in set theory while it is treated with occasional awareness in other branches of classical mathematics.

In constructive mathematics, initiated by Brouwer [7] and developed, among others, by Heyting [31], Bishop [6], Troelstra [61, 62], and Martin-Löf [47], the role of AC is quite schizophrenic. Inspecting its formulation with total relations and choice functions, it states:

$$\mathsf{AC}_{X,Y} \coloneqq \forall R : X \to Y \to \mathbb{P}.\ (\forall x : X.\ \exists y : Y.\ R\ x\ y) \to \exists f : X \to Y.\ \forall x.\ R\ x\ (f\ x)$$

Through the informative status of existential quantification in constructive mathematics, a totality proof for R is nothing but a function that on input x yields a matching y with $R\ x\ y$, so f is definable by projection. It is this tautological reading which Bishop expressed in his infamous quote *"A choice function exists in constructive mathematics, because a choice is implied by the very meaning of existence"* [6].

However, there is a strikingly simple argument used by Diaconescu [13] as well as Goodman and Myhill [28] to show that AC implies the non-constructive excluded middle (LEM), stating that for all propositions $P \vee \neg P$ holds, as soon as propositions are considered extensional: as it is always possible to define a surjection $f : \{0,1\} \to \{P, \top\}$ by setting $f\,0 \coloneqq P$ and $f\,1 \coloneqq \top$, using AC for $R\,Q\,n \coloneqq Q = f\,n$ yields an injection $g : \{P, \top\} \to \{0,1\}$ and one can suddenly decide $P = \top$, and thus P, by testing whether $g\,P = g\,\top$. Here, the crucial use of extensionality is to derive $P = \top$ from P, forcing the choice function to treat the totality proof of R as irrelevant. Because of this circumstance, in a note dedicated to debunking Bishop's unfortunate conflation, Martin-Löf concludes: *"Thus the problem with Zermelo's axiom of choice is not the existence of the choice function but its extensionality, and this is not visible within an extensional framework, like Zermelo-Fraenkel set theory, where all functions are by definition extensional"* [49].

Due to this argument, the actual, i.e. extensional, axiom of choice is a constructive taboo and therefore absent in any foundation for constructive mathematics. Still, some of its fragments like dependent (DC), countable ($\mathsf{AC}_{\mathbb{N}}$), unique (AC!), or quantifier-free choice (QF-AC) are present in the most common base systems, see the following table for an overview. Given this obscure situation, Richman argues that countable choice is a blind spot in constructive mathematics and favoured *"constructive mathematics without choice"* [59].

System	*ext*	*typed*	AC	DC	$\mathsf{AC}_{\mathbb{N},Y}$	$\mathsf{AC}_{\mathbb{N},\mathbb{N}}$	$\mathsf{AC!}_{X,Y}$	$\mathsf{AC!}_{\mathbb{N},\mathbb{N}}$	$\mathsf{QF\text{-}AC}_{\mathbb{N},\mathbb{N}}$
MF [46, 45]	2 levels	yes	×	×	×	×	×	×	×
CIC [12, 11, 53]	no	yes	×	×	×	×	×	×	× / Thm.
EL [62, 66]	yes	no	×	×	×	×	×	×	Axiom
M [40]	yes	no	×	×	×	×	×	Axiom	Cor.
IZF [52]	yes	no	×	×	×	×	Thm.	Cor.	Cor.
HoTT [63]	yes	yes	×	×	×	×	Thm.	Cor.	Cor.
$\mathbf{T}_0$ [15]	no	no	×	×	Thm.	Cor.	×	Cor.	Cor.
MLTT [48]	no	no	Thm.	Cor.	Cor.	Cor.	Cor.	Cor.	Cor.

In fact, in many settings even AC!, stating that every total *and functional* relation contains a function remains a blind spot, for instance collapsing hierarchies of sub-classical principles [2] used in constructive reverse mathematics [36] or making the axioms for synthetic computability incompatible with classical logic [17]. So even without an actual choice being made, AC! guarantees the existence of unwanted functions blurring the fine lens of constructive reasoning, or freely translated from Sartre: *"not choosing is still a choice"*.[1] We therefore advocate a consequential extension of Richman's programme in systematically investigating *"constructive mathematics without* any *choice"*, including all its inherent components of function existence and extensionality principles.

In order to carry out such an investigation, set-theoretic settings like **IZF** are ruled out because of set extensionality as central axiom and unique choice following directly from the definition of functions as total and functional relations. On the type-theoretic side, in pure **MLTT** the extensional AC is indirect to express while **HoTT** inherits extensionality from the univalence axioms and also proves unique choice by the usage of homotopy propositions. We therefore resort to **CIC**, which in its minimal core does not even prove QF-AC and is readily implemented in the Rocq prover [60], allowing us to codify the encyclopedic study of reported logical equivalences and decompositions given in this paper.

Apart from these foundational considerations, we revisit and refine three well-known results of constructive mathematics. First, we observe that the usual argument that AC implies LEM as explained above does not use full AC but only the corollary that every surjection $f : X \to Y$ gives rise to an injection $g : Y \to X$, not necessarily inverting f. In set theory, this is known as the *partition principle* and it is a long-standing open question whether or not it has the full strength of AC [32]. Our observation refines the presentation of Werner [68], remarking that in a type-theoretic setting, assuming proof irrelevance (PI) is enough to replay the argument up to decidability of equality (ELEM), with LEM then following from a weak formulation of propositional extensionality. Secondly, we establish that Coquand's result that LEM implies PI in CIC [11], i.e. that classical logic is incompatible with a proof relevant perspective, already follows from ELEM (even restricted to equalities of propositions) via a careful analysis of Barbanera and Berardi's simplification of Coquand's argument [3], itself based on the Girard-Hurkens paradox [26, 35]. As in an intensional setting ELEM seems to be a much weaker assumption than LEM, this result thus proves that determinable equality is already incompatible with proof relevance. Finally, we prove that function inversion principles, such that every injection has a left inverse, or the Cantor-Bernstein theorem stating that two bi-directional injections induce a bijection, not only require LEM, but also AC!. These observations suggest that in our minimalistic setting, these inversion principles are yet another instance of a hidden usage of choice fragments.

[1] "Je peux toujours choisir, mais je dois savoir que si je ne choisis pas, je choisis encore.", Jean-Paul Sartre, L'existentialisme est un humanisme, p. 73, Les Éditions Nagel, Pensées, 1966

Furthermore, and partially connected to these well-known theorems, we also provide three novel technical perspectives. First, we report that (in a suitably extensional setting) LEM exactly has the strength of $\mathsf{AC}_{X,\mathbb{B}}$ for arbitrary X and therefore can be visibly expressed as a choice principle, but also as an ordering principle, stating that every X can be endowed with an irreflexive relation with at most one minimum. Secondly, we emphasise the role of ELEM as weaker substitute for LEM in an intensional setting, for instance realised in syntactic models that are far from validating LEM. Finally, we study QF-AC in connection to type-theoretic principles such as recursion on accessibility proofs allowed by large elimination.

Figure 1 summarises the main implications between the principles studied in this paper, with side conditions like extensionality or proof irrelevance noted on the edges.

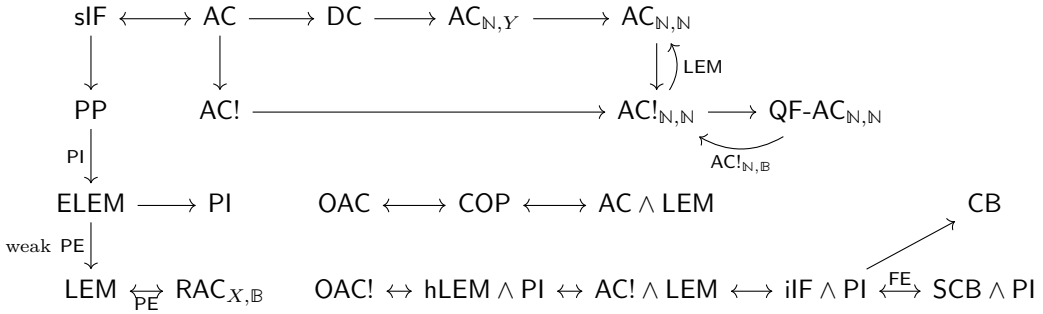

Figure 1 Main implications between principles.

In summary, we consider the contributions of this paper as follows:

1. We provide an atlas of subtle results concerning weak fragments of (unique) choice principles and their interplay with extensionality axioms, organising the somewhat obscure situation of results covered in the literature.
2. We establish the aforementioned concrete theorem refinements and technical perspectives, clarifying some meta-theoretical properties of the constructive type theories underlying modern proof assistants.
3. We mechanise the whole paper in Rocq 9.1 in ∼2k LoC, hyperlinked with the text, giving full detail on informally explained proofs and available as a base for future developments, see `https://github.com/inria-cambium/not-choosing-is-still-a-choice/`.

2 Preliminaries

We work in a minimal variant of the Calculus of Inductive Constructions (**CIC**, [12, 11, 53]). Concretely, like **CIC**, it has a hierarchy of predicative type universes $\mathbb{T}_i$, where we omit the index i, an impredicate universe of propositions $\mathbb{P}$, and inductive types in all universes. We use inductive types $\mathbb{B}$ with constructors true and false, $\mathbb{1}$ with $\star$ as well as $\mathbb{N}$. We use $\langle n, m\rangle$ as Cantor pairing on $\mathbb{N}$. We use the notation $\forall$ and $\to$ for both propositions and types, $\wedge, \vee, \exists$ for propositions where the constructors of $\vee$ are $\mathsf{left}_\vee$ and $\mathsf{right}_\vee$, and $\times, +, \Sigma$ for types with constructors (a, b) for $\times$, $\exists$ and Σ, and inl, inr for $+$. We also use inductive equality $x = y$ with sole constructor $\mathsf{refl} : x = x$. Crucially, *minimal* **CIC** restricts case analysis on proofs, i.e. $H : P : \mathbb{P}$ in a $\mathbb{T}$ context. They are only allowed for falsity and equality, i.e.:

$$\forall T : \mathbb{T}.\ \bot \to T \qquad \forall T : X \to \mathbb{T}.\ T\,x \to x = y \to T\,y$$

Only Facts 8 and 24 rely on the latter, meaning most results would hold for **CIC** with the proof-irrelevant universe `SProp` [25], where however the proof irrelevance axiom would become invisible. For all other propositional types, induction and elimination principles are

only provable in the form $T : X \to \cdots \to Y \to \mathbb{P}$. This difference between propositions and types is most notable for disjunction and sums. For $P \vee Q$, we have

$$\forall R : \mathbb{P}.\ (P \to R) \to (Q \to R) \to P \vee Q \to R$$

but replacing R by an arbitrary type T is not permitted: it would allow proving that there are different proofs of a disjunction (and it would, in general, render the theory inconsistent due to impredicativity). However, for computational sums $X + Y$, one can prove:

$$T : X + Y \to \mathbb{T}.\ (\forall x.\ T\,(\mathsf{inl}\ x)) \to (\forall y.\ T\,(\mathsf{inr}\ y)) \to \forall s : X + Y.\ T\,s$$

A similar situation arises for existentials ($\exists$) and their computational counterpart Σ. For $\exists x : X.\ P\,x$ we have only propositional elimination and for $\Sigma x : X.\ P\,x$ a stronger variant:

$$\forall R : \mathbb{P}.\ (\forall x.\ P\,x \to R) \to (\exists x.\ P\,x) \to R$$
$$\forall R : (\Sigma x : X.\ P\,x) \to \mathbb{T}.\ (\forall x (h : P\,x) \to R\,(x, h)) \to \forall (s : \Sigma x.\ P\,x).\ R\,s$$

In fact, $\exists$ can be defined in terms of Σ using an inductively defined inhabitation predicate $\|\cdot\| : \mathbb{T} \to \mathbb{P}$ with only rule $X \to \|X\|$. Conversely, inhabitation of X can be defined as $\exists x.\ \top$.

Propositions a priori behave as other types, meaning in particular they are by default neither extensional nor proof-irrelevant. Both can however be consistently assumed:

$$\mathsf{PE} := \forall P\,Q : \mathbb{P}.\ (P \leftrightarrow Q) \to P = Q \qquad \mathsf{PI} := \forall P : \mathbb{P}.\ \forall H\,H' : P.\ H = H'$$

Fact 1. *If $P \to P = \top$, then P has unique proofs. So in particular,* PE *implies* PI.

Proof. Assume $H, H' : P$, then by $P \to P = \top$ we have $H, H' : \top$ and therefore $H = H'$. ◀

Further extensionality axioms one can impose concern functions (FE) and classes (CE):

$$\mathsf{FE} := \forall X\,Y.\ \forall f\,g : X \to Y.\ (\forall x.\ f\ x = g\ x) \to f = g$$
$$\mathsf{CE} := \forall X.\ \forall P\,Q : X \to \mathbb{P}.\ (\forall x.\ P\ x \leftrightarrow Q\ x) \to P = Q$$

Fact 2. PE *and* FE *together imply* CE *and* CE *implies* PE.

Proof. Using PE we turn $(\forall x.\ P\ x \leftrightarrow Q\ x)$ into $(\forall x.\ P\ x = Q\ x)$, hence $P = Q$ using FE. ◀

Finally, we distinguish the following variants of the excluded middle:

$$\mathsf{LEM} := \forall P : \mathbb{P}.\ P \vee \neg P \qquad \mathsf{ELEM} := \forall X : \mathbb{T}.\ \forall x\,y : X.\ x = y \vee x \neq y$$

For LEM, we speak of P being *definite* and reserve *decidable* for sums $P + \neg P$ or functions $f : X \to \mathbb{B}$ such that $\forall x.\ P\ x \leftrightarrow f\ x = \mathsf{true}$. In foundations with unique choice, the two are equivalent – other authors thus use the word decidable already for the propositional notion.

3 The intensional axiom of choice

The *intensional axiom of choice* relativised to types X and Y is stated as

$$\mathsf{AC}_{X,Y} := \forall R : X \to Y \to \mathbb{P}.\ (\forall x : X.\ \exists y : Y.\ R\ x\ y) \to \exists f : X \to Y.\ \forall x.\ R\ x\ (f\ x).$$

"Intensional" refers to the fact that there is no guarantee that extensionally equal relations or different totality proofs will yield the same choice function. We write AC for $\forall X\,Y.\ \mathsf{AC}_{X,Y}$ and proceed similarly with other principles when there is no ambiguity. In the system we

work in, AC is not provable, due to the nature of existential quantifiers discussed in the preliminaries. To recover Bishop's axiom of choice, one can state AC using Σ instead of $\exists$.

$$\mathsf{AC}\Sigma_{X,Y} \coloneqq \forall R : X \to Y \to \mathbb{P}.\ (\forall x : X.\ \Sigma y : Y.\ R\ x\ y) \to \Sigma f : X \to Y.\ \forall x : X.\ R\ x\ (f\ x).$$

ACΣ is a direct consequence of the strong elimination rule available for Σ. Calling ACΣ a choice theorem is thus a misleading linguistic abuse: by virtue of the Bishopian motto, there is no choice involved, the input function actually produces witnesses.

Fact 3. ACΣ *holds.*

Proof. Let $h : \forall x : X.\ \Sigma y : Y.\ R\ x\ y$. Then we can define $f\,x \coloneqq \pi_1\,(h\,x)$. ◀

We now review and relate other different standard variants of intensional choice-related principles. As expected, AC is equivalent to the existence of a right inverse to any surjective function. We define surjectivity in the usual way:

$$\mathsf{surjective}\ (f : X \to Y) \coloneqq \forall y.\ \exists x.\ f\,x = y$$

Fact 4. AC *is equivalent to* sIF*, with:*

$$\mathsf{sIF}_{X,Y} \coloneqq \forall f : X \to Y.\ \mathsf{surjective}\ f \to \exists i : Y \to X.\ \forall y.\ f\ (i\ y) = y$$

Proof. Only the converse direction is interesting: If R is total, then $f : (\Sigma(x, y).\ R\,x\,y) \to X$ defined by projection is surjective, so its right inverse provided by sIF is a choice function. ◀

Surjectivity can also be stated as being an epimorphism, in the category of types:

$$\mathsf{epi}\ (f : X \to Y) \coloneqq \forall Z.\ \forall (g_1\ g_2 : Y \to Z).\ (\forall x.\ g_1\,(f\,x) = g_2\,(f\,x)) \to \forall y.\ g_1\,y = g_2\,y$$

Fact 5. *A function is* surjective *iff it is* epi*. The direction from right to left requires* PE.

Note that we state epi up to pointwise equality on functions. Using equality would require FE for the direction from left to right. There are alternative definitions of epi in **HoTT**, with proof-relevant equality, see [8]. We omit them, since they can be proved equivalent under the (consistent) assumption of PI in our setting.

Note that later in this text, we will also consider the following corollary of sIF.

Fact 6. $\mathsf{sIF}_{X,Y}$ *implies* $\mathsf{PP}_{X,Y}$*, with:*

$$\mathsf{PP}_{X,Y} \coloneqq \forall f : X \to Y.\ \mathsf{surjective}\ f \to \exists i : Y \to X.\ \mathsf{injective}\ i$$

We next observe that AC is equivalent to the type of description operators being inhabited:

Fact 7. AC *is equivalent to* Desc*, with:*

$$\mathsf{Desc} \coloneqq \|\forall X.\ \forall P : X \to \mathbb{P}.\ (\exists x.\ P\ x) \to (\Sigma x\ .P\ x)\|$$

Proof. Use AC with the relation

$$R\,((X, P, (x, p_x)) : \Sigma X : \mathbb{T}.\Sigma P : X \to \mathbb{P}.\ \exists x.Px)\,((X', x') : \Sigma X' : \mathbb{T}.x') \coloneqq \exists e : X = X'.P\ x'.$$

The predicate application is well-typed only under the assumed equality e, we omit this technicality. The rest is straightforward. ◀

Now it is trivial to deduce that AC is equivalent to its dependent version, a version with an operator for the choice function, and to the commutation of inhabitation with ∀:

Fact 8. AC *is equivalent to the following:*

1. ACdep, *with* $\mathsf{ACdep}_{X,Y} \coloneqq \forall(R : \forall x : X.\ Y\ x \to \mathbb{P}).\ (\forall x.\ \exists y.\ R\ x\ y) \to \exists(f : \forall x.\ Y\ x).\forall x.\ R\ x\ (f\ x)$
2. ACop, *with:* $\mathsf{ACop} \coloneqq \|\forall X\ Y.\ \forall R : X \to Y \to \mathbb{P}.\ (\forall x.\ \exists y.\ R\ x\ y) \to \Sigma f : X \to Y.\ \forall x.\ R\ x\ (f\ x)\|$
3. ACinhab, *with:* $\mathsf{ACinhab} \coloneqq \forall X\ (Y : X \to \mathbb{T}).\ (\forall x.\ \|Y\ x\|) \to \|\forall x.\ Y\ x\|$

To conclude this collection of folklore facts, observe that the following omniscient form OAC of the axiom of choice comes at odds with constructive foundations:

$$\mathsf{OAC}_{X,Y} \coloneqq \forall R : X \to Y \to \mathbb{P}.\ \exists f : X \to Y.\ (\forall x.\ \exists y.\ R\ x\ y) \to \forall x.\ R\ x\ (f\ x)$$

This formulation indeed inlines an instance of the independence of (general) premise principle:

$$\mathsf{IP}_X \coloneqq \forall P : X \to \mathbb{P}.\ \forall Q : \mathbb{P}.\ (Q \to \exists x.\ P\ x) \to \exists x.\ Q \to P\ x$$

Fact 9. *For inhabited* X, $\mathsf{OAC}_{X,Y}$ *is equivalent to the conjunction of* $\mathsf{AC}_{X,Y}$ *and* IP_Y.

Proof. That $\mathsf{OAC}_{X,Y}$ implies $\mathsf{AC}_{X,Y}$ is trivial and for IP_Y pick the relation $R\,x\,y \coloneqq P\,y$. Conversely, given $R : X \to Y \to \mathbb{P}$ use $\mathsf{AC}_{X,Y}$ on $R'\,x\,y \coloneqq (\exists y'.\,R\,x\,y') \to R\,x\,y$. First, R' is total as, given x, showing $\exists y.\,(\exists y'.\,R\,x\,y') \to R\,x\,y$ turns into a tautology on application of IP_Y. Then, a choice function for R' is in particular an omniscient choice function for R. ◀

Let IP abbreviate IP_X for inhabited X, similarly OAC abbreviate $\mathsf{OAC}_{X,Y}$ for inhabited X and Y. In fact, OAC exactly coincides with the conjunction of AC and LEM and therefore qualifies as a more honest choice axiom in the absence of extensionality.

Fact 10. IP *is equivalent to* LEM, *so in particular* OAC *is equivalent to* $\mathsf{AC} \wedge \mathsf{LEM}$.

Proof. With the previous result and the trivial fact that LEM implies IP, it is only necessary to show the converse of the latter. To that end, let $p : \mathbb{P}$ and set $X \coloneqq \mathbb{1} + p$, $P\,(\mathsf{inl}\,\star) \coloneqq \bot$, $P\,(\mathsf{inr}\,H) \coloneqq \top$, and $Q \coloneqq \exists x.\,P\,x$. Since X is inhabited, from IP we obtain $x : X$ such that $Q \to P\,x$. Now if $x = \mathsf{inr}\,H$ for some $H : p$ we immediately conclude p. If $x = \mathsf{inl}\,\star$, then we conclude $\neg p$ as follows: on assumption of $H : p$, note that we can show Q as $P\,(\mathsf{inr}\,H)$ is a tautology, but then $Q \to P\,x$ implies $P\,(\mathsf{inl}\,\star)$, which is a contradiction. ◀

The equivalence of IP and LEM in **CIC** was observed by Kirst and Zeng [38]. It does not hold in first-order systems which lack quantification over universes [67].

Omniscient choice can also be stated more directly as an omniscience principle for types:

Fact 11. OAC *is equivalent to* COP, *with:*

$$\mathsf{COP} \coloneqq \|\forall X : \mathbb{T}.\ X + (X \to \bot)\|$$

We conclude by stating the axiom of dependent choice, a consequence of AC:

Fact 12. AC *implies* DC, *with:*

$$\mathsf{DC} \coloneqq \forall X.\forall R : X \to X \to \mathbb{P}.(\forall x.\ \exists y.\ R\ x\ y) \to \exists f : \mathbb{N} \to X.\forall i.\ R(f\ i)(f\ (i+1))$$

4 The Diaconescu-Goodman-Myhill theorem

Diaconescu's theorem states that the axiom of choice implies the law of excluded middle. The argument underlying Diaconescu's theorem can be given very compactly if working in set theory, as observed by Goodman and Myhill [28]:

For any two sets a, b there is a surjection $\{0,1\} \to \{a,b\}$. Now the axiom of choice yields an injection $f : \{a,b\} \to \{0,1\}$. Because we can constructively determine equality on $\{0,1\}$ and f is an injection, we have $a = b \vee a \neq b$. For an arbitrary formula ϕ, since ϕ holds iff $\{x \mid x = 0 \wedge \phi\} = \{0\}$ under set extensionality, AC implies LEM. Goodman and Myhill do not make explicit that any injection $\{a,b\} \to \{0,1\}$ suffices, but the argument is the same.

In his paper, Diaconescu, working in a topos, actually proves an equivalence, not just an implication. We observe that in the setting above, there is also an equivalence: $a = b \vee a \neq b$ if and only if an injection $\{a,b\} \to \{0,1\}$ exists. To translate this equivalence to our setting, let A be a type and $a, b : A$. For now, we assume that for all $x : A$ either $x = a$ or $x = b$.

Fact 13. *An injection $A \to \mathbb{B}$ proves $a = b \vee a \neq b$.*

Proof. Let f be the injection. If $f\ a = f\ b$ then $a = b$ because f is injective. If $f\ a \neq f\ b$ then $a \neq b$ because $a = b$ would imply $f\ a = f\ b$. ◀

Note this proof does not rely on any axiom, in particular proof irrelevance is not required.

Fact 14. *From $a = b \vee a \neq b$ we can construct an injective, total, functional relation from A to $\mathbb{B}$, and thus by* AC! *(cf. Section 6) an injection.*

Proof. If $a = b$, then a constant relation proves the goal. If $a \neq b$, define R by taking $R\ a\ \mathsf{true}$ and $R\ b\ \mathsf{false}$. It is total and injective by construction, and functional due to $a \neq b$. ◀

Historic proofs of the theorem Arguably, Bishop anticipated a variant of Diaconescu's result [6, p. 58, ex. 2]: "Construct a surjection which is not onto [... where] *not* is to be understood in the sense of [finding Brouwerian counterexamples]. After discussing a real number which is 0 iff Fermat's last theorem holds, he states "Thus it is unlikely that there will ever exist a constructive proof that for every real number $x \geq 0$ either $x > 0$ or $x = 0$. We express this fact by saying that there exists a real number $x \geq 0$ such that it is not true that $x > 0$ or $x = 0$. In much the same way we can construct a real number x such that it is not true that $x \geq 0$ or $\neg(x \geq 0)$." The first principle is LPO, the limited principle of omniscience, the latter is LLPO. We do not see evidence that Bishop anticipated that LEM, i.e. the full principle of omniscience, would follow.

Werner [68] essentially transposed Goodman and Myhill's proof of $a = b \vee a \neq b$ into type theory, only requiring proof irrelevance to build the type $\{a,b\}$ and then derives ELEM, the law of excluded middle for equalities. As in set theory, propositional extensionality then suffices to obtain full LEM because $P{=}\top \leftrightarrow P$.

The **HoTT** book [63] defines the suspension of a proposition P as a higher inductive type S with elements n, s such that $n = s$ if and only if P holds. The remaining argument is the same as in Fact 13. Other authors have different constructions of the suspension: The proof by Werner can be seen as the suspension $\mathsf{susp}\ P \coloneqq \Sigma p : \mathbb{P}.\ p = P \vee p = \top$. Hofmann [33] assumes FE and PE and essentially defines $\mathsf{susp}\ P \coloneqq \Sigma p : \mathbb{B} \to \mathbb{P}.\ \forall b.\ (p\,b \leftrightarrow b = \mathsf{true} \vee P) \vee (p\,b \leftrightarrow b = \mathsf{false} \vee P)$. If P holds, this type only contains the full predicate, up to CE. Abou Samra observes that one can also define $\mathsf{susp}\ P \coloneqq \Sigma p : \mathbb{P} \times \mathbb{P}.\ p = (\top, P) \vee p = (P, \top)$. If P holds, this type only contains $(\top, \top)$ up to PE (and its consequence PI).[2] Maietti works in a two-level type theory with quotients [44], where one can define $\mathsf{susp}\ P \coloneqq \mathbb{B}/P$.

[2] PR against the Rocq stdlib: `https://github.com/rocq-prover/stdlib/pull/214`

Full proof We analyse that a double negated form of PE for provable propositions suffices, and that due to Fact 13 the partition principle suffices.

Fact 15. PP *and* PI *together imply* ELEM.

Proof. Let $a, b : A$. Define $S \coloneqq \Sigma x.\ x = a \vee x = b$ and a surjection $f : \mathbb{B} \to S$ with $f\mathsf{true} \coloneqq (a, \mathsf{left}_\vee\ \mathsf{refl})$ and $f\mathsf{false} \coloneqq (b, \mathsf{right}_\vee\ \mathsf{refl})$: Given an element in S, i.e. (x, h_x) where $h_x : x = a \vee x = b$, by case analysis on h_x we either have true or false mapping to it.

Now PP yields an injection $i : S \to \mathbb{B}$. By Fact 13, equality on S is definite. If $(a, \mathsf{left}_\vee\ \mathsf{refl}) = (b, \mathsf{right}_\vee\ \mathsf{refl})$, then $a = b$. Otherwise, we prove $a \neq b$: let us assume $a = b$, then it suffices to prove $(a, \mathsf{left}_\vee\ \mathsf{refl}) = (b, \mathsf{right}_\vee\ \mathsf{refl})$ which follows using PI.

We also show how the argument looks like without reference to the Fact 13: we can compare the value of $i(a, \mathsf{left}_\vee\ \mathsf{refl})$ and $i(b, \mathsf{right}_\vee\ \mathsf{refl})$ because they are booleans. If they are equal, by injectivity of i, $a = b$. If they are different, we prove $a \neq b$ by assuming $a = b$ and have $i(a, \mathsf{left}_\vee\ \mathsf{refl}) = i(b, \mathsf{right}_\vee\ \mathsf{refl})$ using PI – contradiction. ◀

Note that the proof does not require eliminating equality into $\mathbb{T}$. The use of PI here is crucial to show the equality $\mathsf{left}_\vee\ \mathsf{refl} = \mathsf{right}_\vee\ \mathsf{refl}$ on type $a = a \vee a = a$. By analysing the proof, one can see that a weak form of PI stating $\neg\neg\forall P : \mathbb{P}.\ \neg\neg\forall p_1 p_2 : P.\ \neg\neg(p_1 = p_2)$ suffices. This is precisely the point where using a $\forall\Sigma$ form of surjections or ACΣ would fail: one would need computational access to the case distinction $x = a$ or $x = b$ to show $\forall s : S.\Sigma b.\ \ldots$. Defining $S \coloneqq \Sigma x.\ (x = a) + (x = b)$ with a computational sum works, but at the end of the proof requires proving $\mathsf{inl}\ \mathsf{refl} = \mathsf{inr}\ \mathsf{refl}$ which is simply false.

Fact 16. ELEM *implies* LEM *under a weak form of* PE*:* $\neg\neg\forall P : \mathbb{P}.\ \neg\neg(P \to \neg\neg P = \top)$.

Proof. For P, we determine $P = \top$ via ELEM. If this is the case, we can prove P. If $P \neq \top$, we may assume a proof of P and have to prove a contradiction. This allows unwrapping all double negations in the extensionality assumption, yielding $P = \top$ – a contradiction. ◀

Werner notes "that this "decidability" is restricted to a disjunction in Prop and that it is not possible to build an actual generic decision function" but of course, with the axiom of choice, we could now get $\|\forall P.\ P + \neg P\|$, see e.g. Fact 33.

One can note that the only instance of AC needed is for relations with boolean range, see Fact 28 for a result providing an equivalence of LEM to such a choice principle.

The Well-Ordering Theorem Adding another angle to the connection of AC and LEM, and as it is well-known that in classical set theory, AC is equivalent to the well-ordering theorem (WO), stating that every set can be well-ordered. In the context of Diaconescu's theorem one can ask which fragment of WO has the exact strength of LEM. Swan observed that LEM already follows if every type can be endowed with an irreflexive relation with at most one minimum,[3] see also [37, Fact 10.32]. In fact this condition is equivalent to LEM.

Fact 17. LEM *if and only if every type can be endowed with an irreflexive relation with at most one minimum. The direction from right to left requires* PE.

Proof. Given a type X, first use LEM to test whether X is inhabited or not. If not, then any relation works, if there is $x_0 : X$ define $R\,x\,y \coloneqq x = x_0 \wedge y \neq x_0$. This is clearly irreflexive and every minimum x is equal to x_0: arguing with LEM, if it were $x \neq x_0$ then $R\,x_0\,x$ would hold, contradicting minimality of x.

For the converse, one can show that for $P : \mathbb{P}$, given such a relation on the type $X \coloneqq \Sigma Q.\ \neg\neg(P = Q)$ induces that $\neg\neg P \to P$, see the cited literature for more detail. ◀

[3] https://ncatlab.org/nlab/show/well-ordering+theorem#in_constructive_mathematics

5 Coquand's proof irrelevance theorem

In his seminal article on **CIC**, Coquand proved that LEM implies PI [11, p. 19] for Leibniz equality, exploiting Girard's paradox [26]. Using inductive equality rather than Leibniz equality, we refine his result by showing that assuming ELEM actually suffices, even when restricted to only apply to equalities of propositions. Our proof relies on Hedberg's theorem [30] and Lawvere's theorem [42], which we quickly recall here and otherwise paraphrases the reformulation of Barbanera and Berardi [3].

Theorem 18 (Hedberg). *Every type with definite equality has unique identity proofs.*

Proof. Given $d : \forall x\ y : X.\ (x = y) \vee (x \neq y)$ we define a function $h : \forall x\ y.\ x = y \to x = y$ that for an input proof $e : x = y$ returns the proof obtained from invoking d, only using e to refute the negative outcome $x \neq y$. Then by construction h is constant, i.e. $h\ x\ y\ e = h\ x\ y\ e'$ for all $e, e' : x = y$.

Now in fact every function $g : \forall x\, y.\ x = y \to x = y$ has a left inverse g^{-1} by setting $g^{-1}e \coloneqq \overline{g\ \mathsf{refl}}; e$, where $\mathsf{refl} : x = x$ is the reflexivity proof for x, the symmetry operation is denoted by overline, and the transitivity operation is denoted by semicolon.

But then h is both constant and injective, inducing that proofs $e, e' : x = y$ are identical:

$$e = h^{-1}\,(h\,e) = h^{-1}\,(h\,e') = e' \qquad \blacktriangleleft$$

Note that under FE, this proof applies to every type with double-negation stable equality.

▶ **Theorem 19** (Lawvere). *If there is a surjection $s : X \to (X \to Y)$, then every function $f : Y \to Y$ has a fixed point.*

Proof. For $g : X \to Y$ defined by $g\,x \coloneqq f\,(s\,x\,x)$ we obtain x with $s\,x = g$ by surjectivity of s. Then $y \coloneqq s\,x\,x$ is a fixed point of f because $f\,y = f\,(s\,x\,x) = g\,x = s\,x\,x = y$. ◀

Now our variant of Coquand's result can be stated and proved as follows.

Theorem 20. *If equality of propositions is definite, then* PI *holds.*

Proof. Consider an inductive (or impredicatively encoded) proposition B with two proofs T and F. Note that for PI it is enough to show $T = F$ as then given an arbitrary proposition P with proofs H and H' we can define a function $f : B \to P$ by cases via $f\,T \coloneqq H$ and $f\,F \coloneqq H'$ for which we then observe $H = f\,T = f\,F = H'$.

We define $U : \mathbb{P} \coloneqq \forall P : \mathbb{P}.\ P \to B$ together with $s : U \to (U \to B)$ by $s\,u \coloneqq u\,U$. This function is a surjection, as we can define its inverse $i : (U \to B) \to U$ by cases

$$i\ H\ P \coloneqq \begin{cases} H^e & \text{if } e : U = P \text{ and } H^e \text{ is the transport of } H \text{ along } e \\ \lambda p.F & \text{otherwise (in fact any element of } P \to B \text{ works)} \end{cases}$$

and then show that $s\,(i\,H) = i\,H\,U = H^e = H$, where in the last step we use Hedberg's theorem to argue that $e : U = U$ is refl and therefore transport along e is the identity. But now Lawvere's theorem applied to the surjection $s : U \to (U \to B)$ yields a fixed point for every function $B \to B$, so in particular there is one for the negation defined by $N\,T \coloneqq F$ and $N\,F \coloneqq T$, from which $T = F$ follows as desired. ◀

Note that in fact only the equality of propositions U and P needs to be determined to define i in the above proof, meaning that already a type conditional in the sense of Harper and Mitchell [29] is enough to enable the construction.

▶ **Corollary 21.** ELEM *implies* PI*, so in particular* LEM *implies* PI*.*

6 The axiom of unique choice

The axiom of unique choice, also called axiom of no-choice, states that every *functional* total relation has a choice function. A relation is functional, if $R\ x\ y \to R\ x\ y' \to y = y'$. Note that constructively weaker formulations of functionality are possible, but we do not cover them here, see e.g. foonote 3 in [64].

We define $\exists!y.\ p\ y \coloneqq \exists y.\ p\ y \wedge \forall y'.\ p\ y' \to y = y'$ and

$$\mathsf{AC!}_{X,Y} \coloneqq \forall R : X \to Y \to \mathbb{P}.\ (\forall x.\ \exists!y.\ R\ x\ y) \to \exists f : X \to Y.\ \forall x.\ R\ x\ (f\ x)$$

In HoTT, the axiom is provable as consequence of modeling propositions semantically. It is evident in set theory, because functions are defined as total functional relations. In **CIC**, however, this is not the case: **CIC** can be thought of as a two level type theory, where functions and total functional relations are strictly separate, not even the most basic case $\mathsf{AC!}_{\mathbb{N},\mathbb{B}}$ is provable in **CIC**. The following connections were observed by Moschovakis [50]:

Fact 22. *For any inhabited Z, $\mathsf{AC!}_{X,Z\to Y}$ implies $\mathsf{AC!}_{X,Y}$.*

Proof. Given a total $R : X \to Y \to \mathbb{P}$ and $z_0 : Z$, define $R'\,x\,f \coloneqq (\forall z_1\ z_2.\ f\ z_1 = f\ z_2) \wedge R\,x\ (f\ z_0)$. Whenever $R\,x\,y$, then $R'\,x\,(\lambda z.\ y)$ holds. A choice function $f' : X \to Z \to Y$ for R' can be turned into a choice function $f\,x \coloneqq f'\ x\ z_0$ for R'. ◀

Fact 23. *Under FE, $\mathsf{AC!}_{\mathbb{N},Y}$ implies $\mathsf{AC!}_{\mathbb{N},\mathbb{N}\to Y}$.*

Proof. Let $R : \mathbb{N} \to (\mathbb{N} \to Y) \to \mathbb{P}$ be functional and total. We rely on the pairing function introduced in Section 2 and define $R'\langle n, x\rangle\ y \coloneqq \exists!g.\ R\,n\,g \wedge g\,x = y$. It is straightforward to prove this relation functional and total, where functionality needs functional extensionality. A choice function f' for R' allows defining a choice function for R as $f\,n\,x \coloneqq f'\,\langle n, x\rangle$. ◀

The proof can be generalised to $\mathsf{AC!}_{X,Y} \to \mathsf{AC!}_{X,Z\to Y}$ as long as there is an injective function $X \times Z \to X$.

Similarly to the full axiom of choice, one can state unique choice as unique description operator Iota or in an omniscient form.

Fact 24. $\mathsf{AC!}$ *is equivalent to* Iota*, with:*

$$\mathsf{Iota} \coloneqq \|\forall X.\ \forall P : X \to \mathbb{P}.\ (\exists!x.\ P\ x) \to (\Sigma x.\ p\ x)\|$$

Proof. Same as Fact 7. ◀

Fact 25. *Under PI, $\mathsf{OAC!}$ is equivalent to $\mathsf{AC!} \wedge \mathsf{LEM}$, with*

$$\mathsf{OAC!}_{X,Y} \coloneqq \forall R : X \to Y \to \mathbb{P}.\ \exists f : X \to Y.\ (\forall x.\ \exists!y.\ R\ x\ y) \to \forall x.\ R\ x\ (f\ x)$$

where $\mathsf{OAC!}$ abbreviates $\mathsf{OAC!}_{X,Y}$ for inhabited X and Y.

Proof. The only interesting part is how $\mathsf{OAC!}$ implies LEM, which is essentially the same proof as for OAC. Take $X \coloneqq \mathbb{1}$, $Y \coloneqq \mathbb{1} + (P \vee \neg P)$, $R\ x\ y \coloneqq \mathsf{if}\ y\ \mathsf{is\ inr}\ u\ \mathsf{then}\ \bot\ \mathsf{else}\ \top$. Both X and Y are inhabited. Now we obtain a function $f : \mathbb{1} \to \mathbb{1} + (P \vee \neg P)$. If f returns the right part of the sum, $P \vee \neg P$ holds. If f returns the left part, we prove $\neg P$, i.e. we assume P and prove $\bot$ by the equation for f and showing that there is a unique element y in $\mathbb{1} + (P \vee \neg P)$ such that $\mathsf{if}\ y\ \mathsf{is\ inr}\ u\ \mathsf{then}\ \bot\ \mathsf{else}\ \top$. ◀

The axiom of countable choice is $\mathsf{AC}_{\mathbb{N},Y}$ for any Y. The special case $\mathsf{AC}_{\mathbb{N},\mathbb{N}}$ is *provable* in **ZF** set theory and thus might be used without recognising it as a choice principle:

Fact 26. *Under* LEM*,* $\mathsf{AC!}_{\mathbb{N},\mathbb{N}}$ *implies* $\mathsf{AC}_{\mathbb{N},\mathbb{N}}$*.*

Proof. Let $R : \mathbb{N} \to \mathbb{N} \to \mathbb{P}$ be total. Define $R'\ x\ y \coloneqq R\ x\ y \wedge \forall y' < y.\ \neg R\ x\ y$. R' is functional and under LEM total. A choice function for R' is also a choice function for R. ◀

Relational choice The axiom of choice can be reformulated to not imply AC! anymore, by stating that every total relations contains a total functional subrelation:

$$\mathsf{RAC}_{X,Y} \coloneqq \forall R : X \to Y \to \mathbb{P}.\ (\forall x.\ \exists y.\ R\ x\ y) \to \exists S \subseteq R.\ \forall x.\ \exists! y.\ S\ x\ y$$

Fact 27. $\mathsf{AC}_{X,Y}$ *is equivalent to* $\mathsf{RAC}_{X,Y} \wedge \mathsf{AC!}_{X,Y}$*.*

We can now identify LEM with a particular form of RAC, provided enought extensionality:

Fact 28. *Given* PE*,* LEM *is equivalent to* $\mathsf{RAC}_{X,\mathbb{B}}$ *for all* X*.*

Proof. Given $R : X \to \mathbb{B} \to \mathbb{P}$, $R'xb \coloneqq$ if b then $R\,x\,\mathsf{true}$ else $\neg R\,x\,\mathsf{true}$ is a subrelation, total, and functional using LEM.

The converse direction is another variant of Diaconescu's theorem, observing that it suffices to have a total functional injective relation rather than a function. ◀

The arithmetical hierarchy The arithmetical hierarchy was introduced by Kleene [39] and Mostowski [51]. While it is related to models of computation, it can be replicated in any foundation for constructive mathematics as follows:

$$\frac{f : \mathbb{N}^k \to \mathbb{B}}{\Sigma^k_0(\lambda \vec{n}.\ f\ \vec{n} = \mathsf{true})} \qquad \frac{f : \mathbb{N}^k \to \mathbb{B}}{\Pi^k_0(\lambda \vec{n}.\ f\ \vec{n} = \mathsf{true})} \qquad \frac{\Pi^{k+1}_n p}{\Sigma^k_{n+1}(\lambda \vec{n}.\ \exists x.\ p\,(x :: \vec{n}))} \qquad \frac{\Sigma^{k+1}_n p}{\Pi^k_{n+1}(\lambda \vec{n}.\ \forall x.\ p\,(x :: \vec{n}))}$$

Equivalently, one can ask that the 0 level is given by a quantifier-free predicate. However, with function variables f, the given formula is quantifier-free, and quantifier-free formulas remain decided by a function $f : \mathbb{N}^k \to \mathbb{B}$.

Akama, Berardi, Hayashi, and Kohlenbach [2] introduce the arithmetical hierarchy of the law of excluded middle over Heything arithmetic: Σ^k_n-LEM states that LEM is true for any Σ^k_n-predicate, and similarly fo Π^k_n-LEM. This hierarchy was studied further by Fujiwara and colleagues over **EL** e.g. in [23, 24], and also in **CIC** [20].

We note that studying these axioms over a weak base theory is crucial: **HA** and **EL** in particular do not prove the axiom of unique choice, because otherwise:

Fact 29. *With* $\mathsf{AC!}_{\mathbb{N},\mathbb{B}}$*, the arithmetical hierarchy of* LEM *collapses to level* 1*:*

$$\mathsf{AC!}_{\mathbb{N},\mathbb{B}} \to \Sigma^0_1\text{-LEM} \to (\forall n.\ \forall p.\ \Sigma^0_n p \leftrightarrow \Sigma^0_0 p) \wedge \forall n.\ \Sigma^0_n\text{-LEM}$$

Consequently, Π^0_2-LEM*,* Σ^0_2-DNE*, and* Π^0_2-DNE *have the analogous effect.*

Proof. With $\mathsf{AC}_{\mathbb{N},\mathbb{B}}$ and Σ^0_1-LEM we can prove that every Σ^0_1 predicate is decidable, and thus in Σ^0_0. However, then $\Sigma^0_1 = \Sigma^0_0$. By induction, every Σ^1_n becomes Σ^0_0. ◀

We believe that the arithmetical hierarchy of double negation elimination stays intact.

Church's thesis Church's thesis CT states that all functions $\mathbb{N} \to \mathbb{N}$ are computable. In particular, it allows proving that there is a Π^0_1 predicate K on $\mathbb{N}$ which is not decided by any function $\mathbb{N} \to \mathbb{B}$. Unique choice together with classical logic renders CT invalid in **CIC**, but both components seem to be necessary. The proof is straightforward:

▶ **Fact 30.** Π^0_1-LEM *and* Π^0_1-$\mathsf{AC!}_{\mathbb{N},\mathbb{B}}$ *prove that every* Π^0_1 *predicate on* $\mathbb{N}$ *is decided by a function* $\mathbb{N} \to \mathbb{B}$ *and thus disprove* CT*.*

7 Invertibility of functions

We have already seen that the inversion principle for surjections sIF is equivalent to the axiom of choice, and have discussed PP in the context of Diaconescu's theorem. We now turn to their duals: the inversion principle for injections iIF and the co partition principle.

$$\mathsf{iIF} := \forall X\ Y.\ \|X\| \to \forall i : X \to Y.\ \mathsf{injective}\ i \to \exists s : Y \to X.\ \forall x.\ s\,(i\,x) = x$$

$$\mathsf{coPP} := \forall X\ Y.\ \|X\| \to \forall i : X \to Y.\ \mathsf{injective}\ i \to \exists s : Y \to X.\ \mathsf{surjective}\ s$$

The following was observed by Pradic and Brown [55] in their paper on the reverse analysis of the Cantor Bernstein theorem. We slightly refine their proof to explicitly use PI:

Lemma 31. $\mathsf{coPP} \wedge \mathsf{PI} \to \mathsf{LEM}$

Proof. Define $i : P + \mathbb{1} \to \mathbb{B}$ which sends proofs of P to true and the element $\star$ of $\mathbb{1}$ to false. This is clearly an injection assuming PI. Now coPP yields a surjection $s : \mathbb{B} \to P + \mathbb{1}$. We can now distinguish cases which side of the sum s maps to. The only interesting case is $g\ \mathsf{true} = g\ \mathsf{false} = \mathsf{inr}\ \star$, which proves $\neg P$ by surjectivity of s. ◀

We observe that this idea can be continued to obtain an equivalence:

Fact 32. $\mathsf{iIF} \wedge \mathsf{PI} \leftrightarrow \mathsf{LEM} \wedge \mathsf{AC!}$

Proof. First, iIF ∧ PI implies LEM via Lemma 31 and LEM implies PI via Theorem 20. So we need to prove that (1) iIF ∧ PI implies AC!, and that (2) LEM ∧ AC! implies iIF.

For (1), we may assume LEM as well. To prove AC!, assume $\forall x.\exists! y.\ Rxy$. First, we use LEM to do a case analysis on whether X is inhabited. If not, we are done, if so, we may invoke iIF: We define an injection $i : (\Sigma x : X.\Sigma y : Y.\ Rxy) \to X$ by projection. By iIF we obtain an inverting $s : X \to (\Sigma x : X.\Sigma y : Y.\ Rxy)$, which allows defining a choice function $f : X \to Y$ by projection on $s\,x$.

For (2), assume an injection $i : X \to Y$ and $x_0 : X$. We define a surjection by applying AC! to the relation which maps y to x if $ix = y$ and to x_0 otherwise. ◀

Lastly, we discuss a variant of LEM as found in univalent mathematics. We observe that formulating it in **CIC** under proof irrelevance also is exactly a bundle of the usual law of excluded middle with AC!:

$$\mathsf{hprop}\ T := \forall x\ y : T.\ x = y \qquad \mathsf{hLEM} := \|\forall T : \mathbb{T}.\ \mathsf{hprop}\ T \to T + \neg T\|$$

Fact 33. $\mathsf{hLEM} \wedge \mathsf{PI} \leftrightarrow \mathsf{LEM} \wedge \mathsf{AC!}$*, so in the presence of* PI *we have* $\mathsf{hLEM} \leftrightarrow \mathsf{iIF}$.

Proof. Obviously, hLEM implies LEM under PI. Morevover, that hLEM implies AC! is by observing that $\Sigma y.\ R\,x\,y$ is an hprop under PI if R is functional.

The other direction makes use of Iota from Fact 24. Because we have to prove a truncation, it allows us to assume $\forall X.\ \forall P : X \to \mathbb{P}.\ (\exists! x.\ Px) \to \Sigma x.\ Px$. We can now prove $\forall T : \mathbb{T}.\ \mathsf{hprop}\ T \to T + \neg T$. Given T with hprop T, we can use the assumption by proving $\exists! y : T + \neg T.\ \top$ to obtain $T + \neg T$ as needed. ◀

8 Cantor-Bernstein theorem

The Cantor-Bernstein theorem CB states that if there are injections between two types in either direction, then the types are in bijection (where we denote bijectivity with bij):

$$\mathsf{CB} := \forall XY.\ \forall f : X \to Y.\ \forall g : Y \to X.\ \mathsf{injective}\ f \to \mathsf{injective}\ g \to \exists h : X \to Y.\ \mathsf{bij}\ h$$

Pradic and Brown showed that CB implies excluded middle [55] while Escardó has shown that over MLTT, CB is equivalent to hLEM [14]. We observe that in our setting, under enough extensionality, CB implies LEM, and a computational form of CB dubbed SCB à la Escardó is exactly equivalent to LEM ∧ AC!.

Theorem 34 (Cantor-Bernstein)**.** LEM *and* AC! *together imply* CB.

For the backward direction, we record an intermediate fact we find worthwhile stating as it does not yet rely on strong extensionality principles but just proof irrelevance.

Fact 35. *Given* PI*,* CB *implies that every predicate* $P : \mathbb{N} \to \mathbb{P}$ *is enumerable, that is, there is a function* $f : \mathbb{N} \to \mathbb{N}$ *such that* Px *iff* $\exists n.\, f\, n = x + 1$*.*

Proof. Note that there are injections $\mathbb{N} \to \mathbb{N} + \Sigma x.\, P\, x$ using the inl constructor and $(\mathbb{N} + \Sigma n.\, P\, n) \to \mathbb{N}$ mapping into the even and odd numbers, respectively. Then by CB there in particular is a surjection $s : \mathbb{N} \to \mathbb{N} + \Sigma x.\, P\, x$ from which we can define f by $f\, n := 0$ if $s\, n = \mathsf{inl}\, x$ for some $x : \mathbb{N}$ and $f\, n := x + 1$ if $s\, n = \mathsf{inr}\, (x, H)$ for some $x : \mathbb{N}$ and $H : P\, x$. ◀

Note that LEM can be de-composed into Kripke's schema, stating that any proposition is Σ^0_1, and Markov's principle, stating that any Σ^0_1 proposition is double negation stable [65]. In particular, in a proof-irrelevant setting, Kripke's schema follows from CB, but it seems to be an open problem whether then CB still implies full LEM. Also note that in the above proof the role of $\mathbb{N}$ can be replaced by any type.

Further analysing [55], the full backward direction can be carried out using function extensionality, we again refer to the Rocq code for the proof:

Fact 36. *Given* PI *and* FE*,* CB *implies* LEM*.*

It seems unlikely that the propositional formulation of CB implies full AC!, as the resulting bijections are shielded by existential quantifiers. We therefore formulate a stronger version that provides computational access to the bijections. For this version, by combining previous results we obtain the following characterisation:

Fact 37. *Given* FE*,* SCB ∧ PI *is equivalent to* LEM ∧ AC!*, with:*

$$\mathsf{SCB} := \|\forall XY.\, \forall f : X \to Y.\, \forall g : Y \to X.\, \mathsf{injective}\ f \to \mathsf{injective}\ g \to \Sigma h : X \to Y.\, \mathsf{bij}\ h\|$$

Proof. That SCB together with PI and FE implies LEM follows from the previous fact and that AC! follows works as in Fact 32. Conversely, LEM alone implies PI by Theorem 19 and that LEM and AC! together actually establish SCB is a simple refinement of Fact 34. ◀

For a constructive variant of CB on countable types see Forster, Jahn, and Smolka [19] and for how much this generalises to other types see Pradic [56].

9 Quantifier-free countable choice

Quantifier-free choice for relations on $\mathbb{N}$ is the only choice principle assumed in systems like **EL**. It is provable in **CIC**, but not our minimal version, and not in **CoC**, i.e. **CIC** without inductive types. We state it, equivalently, using boolean relations, which works in all systems with function symbols:

$$\mathsf{DAC}_{X,Y} := \forall r : X \to Y \to \mathbb{B}.\, (\forall x. \exists y.\, r\, x\, y = \mathsf{true}) \to \exists f : X \to Y. \forall x.\, r\, x\, (f\, x) = \mathsf{true}$$

As long as the codomain Y of the relations is finite, DAC is provable, for instance:

Fact 38. $\mathsf{DAC}_{X,\mathbb{B}}$ *is provable for every* X*.*

Proof. Given $r : X \to \mathbb{B} \to \mathbb{B}$, define $f\,x \coloneqq r\,x\,\mathsf{true}$. ◀

Moreover, DAC is entangled with unique choice, as observed by Vafeiadou [64]:

Fact 39. *For any type* X *with decidable equality, together with functions* $\mathsf{unpair} : X \to X \times X$ *and* $\mathsf{pair} : X \times X \to X$ *such that* $\forall a : X \times X, \mathsf{unpair}\ (\mathsf{pair}\ a) = a$*, we have:*

$$\mathsf{AC!}_{X,\mathbb{B}} \wedge \mathsf{DAC}_{X,X} \to \mathsf{AC!}_{X,X}$$

Proof. Given R, define $R' \coloneqq \lambda(x : X)(b : \mathbb{B}).\,\mathtt{let}\ (y, z) \coloneqq \mathsf{unpair}\ x\ \mathtt{in}\ (b = \mathtt{true} \leftrightarrow R\ y\ z)$. Using $\forall x.\exists!y.\ R\,x\,y$ we can show $\forall z_1 z_2.\ R\,z_1\,z_2 \vee \neg R\,z_1\,z_2$, and thus $\forall x.\,\exists!(b : \mathbb{B}).\,R'\ x\ b$. Obtaining f from $\mathsf{AC!}_{X,\mathbb{B}}$ we define $R'' : X \to X \to \mathbb{B} \coloneqq \lambda xy.\ f\ (\mathsf{pair}\ (x, y))$ and can prove $\forall x.\,\exists y.\,R''\ x\ y = \mathsf{true}$, from which we can use $\mathsf{DAC}_{X,X}$. This gives us a function $g : X \to X$ that will exactly be the choice function we need to conclude. ◀

An example of a type X as in the fact is $\mathbb{N}$, for which we even get the reverse direction [64].

Fact 40. $\mathsf{AC!}_{\mathbb{N},\mathbb{N}} \to \mathsf{AC!}_{\mathbb{N},\mathbb{B}} \wedge \mathsf{DAC}_{\mathbb{N},\mathbb{N}}$

Proof. Only proving $\mathsf{DAC}_{\mathbb{N},\mathbb{N}}$ is interesting. Assume $\forall x.\,\exists y.\,R\ x\ y = \mathsf{true}$. We define $R' \coloneqq \lambda(x\ y : \mathbb{N}).\,R\ x\ y\ \wedge\ (\forall z.\,z < y \to \neg(R\ x\ z))$. By checking $R\ x\ z$ on all natural numbers z below the y such that $R\,x\,y$, we recover unique existence for R' and conclude. ◀

In full **CIC**, $\mathsf{DAC}_{X,\mathbb{N}}$ can be proved using a computation elimination principle AR_R for the inductively defined predicate of accessible points of relations $R : A \to A \to \mathbb{P}$:

$$\frac{H : \forall y : A.\,R\ y\ x \to \mathsf{Acc}\ y}{\mathsf{AI}\,H : \mathsf{Acc}\ x} \qquad \mathsf{AR}_R \coloneqq \begin{array}{l}\forall T : A \to \mathbb{T}.\,(\forall x.\,(\forall y.\,R\ y\ x \to T\ y) \to T\ x) \to \\ \forall x.\,\mathsf{Acc}\ x \to T\ x.\end{array}$$

Our minimal variant of **CIC** does not prove AR_R. We now show that in fact, DAC is equivalent to AR_R for computationally finitary $R : A \to A \to \mathbb{P}$, meaning there is a function $R_l : A \to A^*$ returning the list of elements below y, i.e. $\forall a\,b.\ R\,a\,b \leftrightarrow a \in R_l\,b$.

Fact 41. *If* AR_R *holds for all computationally finitary* $R : \mathbb{N} \to \mathbb{N} \to \mathbb{P}$*, then* $\forall X.\,\mathsf{DAC}_{X,\mathbb{N}}$*.*

Proof. For $r : X \to \mathbb{N} \to \mathbb{B}$ with totality $\forall x.\,\exists n.\,r\,x\,n = \mathsf{true}$ and some $x : X$, consider the relation $R\,n'\,n \coloneqq r\,x\,n = \mathsf{false} \wedge n' = n + 1$. It is computationally finitary by $R_l\,n \coloneqq [n + 1]$ if $r\,x\,n = \mathsf{false}$, and $R_l\,n \coloneqq []$ otherwise. We prove $r\,x\,(n + k) = \mathsf{true} \to \mathsf{Acc}\,k$ by induction on n, and thus deduce $\mathsf{Acc}\,0$ from the (propositional) existence of n such that $r\,x\,n = \mathsf{true}$.

Note that all reasoning so far was in $\mathbb{P}$, in particular allowing the inspection of the totality proof. Now exploiting AR_R, we can define a function $\forall n.\,\mathsf{Acc}\,n \to \Sigma k.\,r\,x\,k = \mathsf{true}$ in $\mathbb{T}$, to which we can feed the certificate $\mathsf{Acc}\,0$ derived above. By projection, this defines a choice function $f : X \to \mathbb{N}$ that maps any x to its corresponding k. ◀

Note that by virtue of the previous proof, AR_R actually proves the stronger computational principle $\forall f : \mathbb{N} \to \mathbb{B}.\,(\exists n.\,f\,n = \mathsf{true}) \to \Sigma n.\,f\,n = \mathsf{true}$. Conversely, the above instance of DAC is enough to derive AR_R for computationally finitary relations on discrete types.

Fact 42. *Under* PI*, if* A *has decidable equality and* R *is computationally finitary, we have* $(\forall X : \mathbb{T}.\,\mathsf{DAC}_{X,\mathbb{N}}) \to \mathsf{AR}_R$*, and up to* FE *the resulting function fulfills the expected equation.*

Proof. Intuitively, we give a step-indexed function computing the result, show that a step-index n exists from Acc, and compute n using DAC. Details are in Appendix A. ◀

In other contexts, DAC is called Δ^0_1-choice due to the following connection.

Fact 43. *The following are equivalent:*

1. $\mathsf{AC}_{\mathbb{N},\mathbb{N}}$ *for decidable relations (i.e. given by* $r : \mathbb{N} \to \mathbb{N} \to \mathbb{B}$*)*
2. $\mathsf{AC}_{\mathbb{N},\mathbb{N}}$ *for* Σ^0_1*-relations*
3. $\mathsf{AC}_{\mathbb{N},\mathbb{N}}$ *for* Δ^0_1*-relations*

Proof. For (1) to (2), assume $\mathsf{AC}_{\mathbb{N},\mathbb{N}}$ for decidable relations, and let $R : \mathbb{N} \to \mathbb{N} \to \mathbb{P}$ be total and of the form $R\ x\ y := \exists n : \mathbb{N}.\, r\ x\ y\ n = \mathsf{true}$. Totality and Cantor pairing yield $\forall x.\, \exists y.\, \mathtt{let}\ (k, l) := \mathsf{unpair}\ y\ \mathtt{in}\ R'\ x\ k\ l = \mathsf{true}$. We can then use $\mathsf{AC}_{\mathbb{N},\mathbb{N}}$ for decidable relations and conclude. The reverse case is immediate. (2) to (3) is trivial: any Δ^0_1 relations is Σ^0_1. (3) to (1) is also trivial: any decidable relation is Σ^0_0 and thus in particular Δ^0_1. ◀

We discuss provability of AR_R in **CIC** in the next section. Using AR_R and similar techniques to deduce DAC in **CIC** was first employed by Benjamin Werner and Jean-François Monin in the 1990s, without associated publication, but contributed to Rocq's stdlib.[4]

Many results in constructive mathematics crucially rely on the presence of DAC, e.g. in synthetic computability theory [18], studies of Weak König's Lemma [4], to make natural formulations of Markov's principle equivalent [9], to show that μ-recursive or λ-computable relations yield functions [22, 41], or the equivalence of Markov's principle to Post's theorem [62, 21], the latter being one of the first results of constructive reverse mathematics.

The Mathematical Components library [43] actually devotes an entire chapter[5] to the study of types X equipped with decidable equality and a *choice operator* for boolean predicates $P : X \to \mathbb{B}$. It defines an interface for such types, called *choice types*, which exposes in particular a choice function, $\epsilon_X : \forall P : X \to \mathbb{B}.\, (\exists x.\, P\ x = \mathsf{true}) \to X$, such that $P\ (\epsilon_X\ P\ p) = \mathsf{true}$ for a proof of existence $p : \exists x.\, P\ x = \mathsf{true}$, and its uncurried version $c_X : (X \to \mathbb{B}) \to X \to \mathbb{B}$, such that $P\ (c_X\ P\ x_0) = \mathsf{true}$ iff $P\ x_0 = \mathsf{true}$. Both these choice functions pick identical witnesses for pointwise equal boolean predicates. Note however that these functions do not compose well: the availability of such choice functions on types X and Y may not suffice to build a choice function for the type $X \times Y$. The (internal) interface for choice types is thus defined in terms of a slightly stronger operation $\phi_X : (X \to \mathbb{B}) \to \mathbb{N} \to \mathsf{Option}\ X$, which performs a bounded search, using an integer parameter and such that $\phi_X\ P\ n = \mathsf{Some}\ x \to P\ x$. Then, countable types are proved to be choice types, via DAC. The library moreover provides a systematic way to make a countable (resp. choice) type out of any type isomorphic to an instance of a certain gadget data structure, namely a finitely branching tree with inhabitants of given choice types at the leaves, using Gödel-style encodings of sequences of integers. This infrastructure is used to conveniently, albeit constructively mechanise some fundamental results in algebra, such as the construction of field extensions of finite degree over a countable base field [5].

10 On large elimination

An elimination of an element of type I is large if it returns a type. To define a dependent case analysis function on an (inductive) type I one usually needs large elimination on I to define the type of the case analysis.

[4] `https://rocq-prover.org/doc/v9.1/stdlib/Stdlib.Logic.ConstructiveEpsilon.html`
[5] `https://github.com/math-comp/math-comp/blob/master/boot/choice.v`

In this section, we discuss how **CIC** restricts large eliminations if I is a proposition. Historically, Rocq introduced a relaxed criterion for dependent elimination of $\mathbb{P}$-valued inductive types such as Acc to allow classical logic in termination proofs of programs that later can be extracted to OCaml. In general, the elimination principle for some inductive I in a sort s must range over predicates living in the same sort s. This is what prevents proving AC when $\exists$ is defined in $\mathbb{P}$.

This restriction is crucial for various reasons, most notably consistency. In the first versions of Rocq, there were three sorts to choose from, $\mathbb{T}$, $\mathbb{P}$ and the now-defunct impredicative Set. Allowing arbitrary elimination from an impredicative sort to $\mathbb{T}$ leads to an inconsistency, and thus must be forbidden. More recently, Rocq introduced the sort of proof-irrelevant propositions SProp [25] and even abstract sorts [54], and the sort elimination restriction was carried over for the very same reasons.

Nonetheless, if all cross-sort eliminations are forbidden, there is no way to make the various sort hierarchies communicate between one another. Rocq has always allowed eliminating from $\mathbb{T}$ to any sort, but this alone only allows a one-way flowing of communication. This stringent condition is then relaxed in the following way: if an inductive type in $\mathbb{P}$ satisfies the *subsingleton* criterion, it can be eliminated into $\mathbb{T}$. An inductive is a subsingleton when it has at most one constructor, and all arguments to that optional constructor *which are not parameters of the inductive type* are proofs in $\mathbb{P}$. There are three archetypical subsingleton inductive types:

- falsity $\bot$, the type with no constructors;
- equality $=$, the type with a single argument-free constructor but one index;
- accessibility Acc, the type of well-foundedness proofs.

All three types emphasise a distinct aspect of subsingletons. Falsity is about inconsistency, equality about indices and accessibility about the ability to write fixed points. These properties do not need a concrete witness in e.g. a realizability model or when extracting to e.g. OCaml, so these eliminations can be erased away: falsity elimination is a crash, equality elimination an identity cast and accessibility elimination an *a priori* unbounded fixed point. It is conjectured that the three are universal in the sense that other subsingletons can be reduced to them, but we are not aware of a proof.

Technically, the restriction of elimination is implemented in the typing rule of the `match` construct. When eliminating c of inductive type I with parameters q and indices t, with return predicate P in sort s which I is allowed to be eliminated in, and all the branches have appropriate type P applied to the respective constructor, then the case analysis has type $P\ \bar{t}\ c$

$$\frac{\Gamma \vdash c : I\ \bar{q}\ \bar{t} \qquad \Gamma \vdash P : s \qquad [I\ \bar{q} \mid s] \qquad (\Gamma \vdash f_i : \{c_i\ \bar{q}\}^P)_{i=1}^l}{\Gamma \vdash \texttt{match}\ c\ \texttt{as}\ x\ \texttt{in}\ I\ \bar{q}\ \bar{y}\ \texttt{return}\ P\ \bar{y}\ x\ \texttt{with}\ f_1 \mid \ldots \mid f_l\ \texttt{end} : P\ \bar{t}\ c}\ \textsc{match}$$

where $\{c : (I\ \bar{q}\ \bar{t})\}^P \coloneqq P\ \bar{t}\ c$ and $\{c : \forall x{:}T,\, C\}^P \coloneqq \forall x{:}T,\, \{(c\ x) : C\}^P$. The crucial ingredient is the sort elimination predicate $[I\ \bar{q} \mid s]$ given by:

$$\frac{s_1 \in \{\mathsf{Set}, \mathbb{T}_j\} \qquad s_2 \in \mathsf{Sort}}{[I : s_1 \mid I \to s_2]}\ \textsc{Set/Type} \qquad \frac{s \in \{\mathbb{P}\}}{[I : \mathbb{P} \mid I \to s]}\ \textsc{Prop}$$

$$\frac{I \text{ empty or singleton} \qquad s \in \mathsf{Sort}}{[I : \mathbb{P} \mid I \to s]}\ \textsc{Prop-ext} \qquad \frac{[(I\ x : A') \mid B']}{[I : \forall x{:}A,\, A' \mid \forall x{:}A,\, B']}\ \textsc{Prod}$$

The Prop-ext rule is the one allowing large elimination for falsity, equality, and accessibility, because the first is empty and the others are singletons.

However, it is possible to prove a large elimination principle for Acc without a `match` eliminating into $\mathbb{T}$, by delaying the match until after the recursive call, a technique used to prove `Fix` in Rocq's standard library.[6] This hints at the fact that to really forbid large eliminations, one also has to restrict the fixed point typing rule with an elimination predicate as follows:

$$\frac{\Gamma \vdash A : s \qquad \Gamma;\ f : A \vdash t : A \qquad A = \forall A_1 \dots A_n.\ R \qquad A_i = I\ \bar{q}\ \bar{u} \qquad \textit{guarded}(f \text{ at } i \text{ in } t) \qquad [I\ \bar{q} \mid s]}{\Gamma \vdash \texttt{fix}\ \ f : A := t : A}\ \textsc{Fix}$$

As noted, it seems like the discussed three types are the only essential subsingleton types. However, especially historic proofs of DAC have used other types, e.g. the `before_witness` predicate in conjunction with the trick to match only after recursive calls.[7] In general, such predicates can also be expressed as accessibility if needed, meaning that allowing their elimination into $\mathbb{T}$ is not essential.

It is also noteworthy that the subsingleton criterion is *not* valid for the very similar-looking SProp sort in Rocq. The major difference between $\mathbb{P}$ and SProp is that the latter satisfies definitional proof-irrelevance, i.e. any two terms $p, q : P : \mathsf{SProp}$ are convertible. While elimination of the SProp empty type is allowed, the two other subsingleton eliminations lead to metatheoretical issues: Adding equality elimination in a naive way leads to a failure of normalisation [1]. It can be retrieved in a roundabout way, which requires a complete rehauling of the system as an observational type theory [57, 58]. Meanwhile, adding accessibility elimination makes type-checking undecidable in an essential way. By definitional irrelevance, all accessibility proofs can be η-expanded arbitrarily deep, so fixed points defined over well-founded relations will always compute, even in an inconsistent context. As a result, Rocq forbids both eliminations. Lean does allow both, at the price of an incomplete type-checking algorithm and thus breakage of subject reduction. Recent work suggests solutions for both [16].

11 Future work and open questions

To date, we could not find a proof that PP implies AC! but deem it possible. Similarly, in a proof-irrelevant setting, CB yields Kripke's schema, and adding functional extensionality yields LEM, but we do not know whether CB yields LEM without functional extensionality. The dual of CB, stating that bi-directional surjections yield a bijection, trivially follows from PP and CB but seems not to imply another fragment of AC or LEM.

Moreover, it would be interesting to understand whether ELEM is strictly weaker than LEM. If it is, then it would be interesting to see whether the Myhill-Goodman theorem in type theory can be strengthened to yield LEM when just assuming $\mathsf{AC} \wedge \mathsf{PI}$. A candidate for a principle in between is the following propositional variant of Kripke's schema for predicates:

$$\forall X. \forall P : X \to \mathbb{P}. \exists A : X \to \mathbb{N} \to \mathbb{P}.\ (\forall x n.\ Axn \vee \neg Axn) \wedge \forall x.\ Px \leftrightarrow \exists n.\ Axn$$

In particular we are interested in consistency proofs of $\mathsf{AC} \wedge \mathsf{PI} \wedge \mathsf{ELEM} \wedge \mathsf{CT} \wedge \neg\mathsf{LPO}$ or $\mathsf{RAC} \wedge \mathsf{PE} \wedge \mathsf{FE} \wedge \mathsf{PI} \wedge \mathsf{LEM} \wedge \mathsf{CT} \wedge \mathsf{DAC}$ for minimal or full **CIC**, where preliminary variants like $\mathsf{CT} \wedge \mathsf{LEM}$ can already be studied for **EL**.

[6] `https://rocq-prover.org/doc/v9.1/corelib/Corelib.Program.Wf.html#Fix_F_sub`
[7] `https://rocq-prover.org/doc/v9.1/stdlib/Stdlib.Logic.ConstructiveEpsilon.html#before_witness`

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A Proof of Fact 42

Given $a : A$ and $l : A^*$, Let us make a difference between propositional membership of x into l, written $x \in l$ and living in $\mathbb{P}$, and proof-relevant membership x `in` l living in $\mathbb{T}$. Then, to prove Fact 42, we go through the following more general (albeit less readable) lemma.

▶ **Fact 44.** *Let A be a type, not assuming decidable equality on A. Let $R : A \to A \to \mathbb{P}$ be a relation such that $\forall x\, y. \forall (h_1\, h_2 : R\, x\, y).\, h_1 = h_2$. Let us assume a function $R_l : A \to \mathsf{List}\ A$ such that $R\ y\ x \leftrightarrow y$ `in` $(R_l\ x)$. Then we have the following: $(\forall X : \mathbb{T}.\, \mathsf{DAC}_{X,\mathbb{N}}) \to \Sigma \mathcal{F} : \mathsf{AR}_R$.*

Proof. We outline the main steps of the proof and refer the interested reader to the Rocq code. We want to build a function of the following type:

$$\mathcal{F} : (\forall x : A.\, (\forall y : A.\, R\ y\ x \to P\ y) \to P\ x) \to \forall x : A.\, \mathsf{Acc}\ R\ x \to P\ x.$$

From the fact that R is finitely branching, we can build a recursive function that calls itself recursively on all elements of $R_l\ x$. However, since termination of such a function is unclear, we can only define a partial function, with a natural number acting as fuel, and returning $\mathsf{Option}\,(P\,x)$, which is $\mathbb{1} + P\,x$ with explicit constructors None and Some.

$$\mathcal{F}_{\mathsf{par}} : (\forall x : A.\, (\forall y : A.\, R\ y\ x \to P\ y) \to P\ x) \to \forall x : A.\, \mathsf{Acc}\ R\ x \to \mathbb{N} \to \mathsf{Option}\,(P\,x).$$

The whole idea is then to prove that

$$\forall\ H\ x\ H_x.\, \exists (n : \mathbb{N})(p : P\ x).\, \mathcal{F}_{\mathsf{par}}\ H\ x\ H_x\ n = \mathsf{Some}\ p$$

This will be true because $\mathsf{Acc}\,R\,x$ is basically saying that the number of recursive calls will be finite. We will however need to use DAC in the recursive case to go from propositional existence to concrete existence of $m_y : \mathbb{N}$ for every y below x. Finally, since for a given n, $\exists (p : P\ x).\, F\ H\ x\ H_x\ n = \mathsf{Some}\ p$ is decidable, we can use DAC again to extract this natural number and retrieve a total function $\mathcal{F} : \mathsf{AR}_R\ \ A\ R$. ◀

Let us emphasise that the use of proof-relevant x `in` l is required for this proof to go through, as we need to recurse on a proof of x `in` l to compute $\mathcal{F}$. To prove Fact 42, note that decidable equality on A is enough to turn propositional membership $x \in l$ for $l : \mathsf{List}\ A$ into proof-relevant membership x `in` l, meaning we can use Fact 44.

Regarding computations rules of the retrieved function, we have the following.

▶ **Fact 45.** *Assuming function extensionality, $\mathcal{F}$ enjoys the following computation rule:*

$$\forall (H : \forall x : A.\, (\forall y : A.\, R\ y\ x \to P\ y) \to P\ x)(x : A)(H_x : \forall y : A.\, R\ y\ x \to \mathsf{Acc}\ y).$$
$$\mathcal{F}\ H\ x\ (\mathsf{AI}\ H_x) = H\ x\ (\lambda (y : A)(H_y : R\,y\,x).\, \mathcal{F}\ H\ y\ (H_x\ y\ H_y)$$

Function extensionality is needed because the hypothesis H may otherwise distinguish between two extensionally equal arguments.