A zoo of continuity properties in constructive type theory

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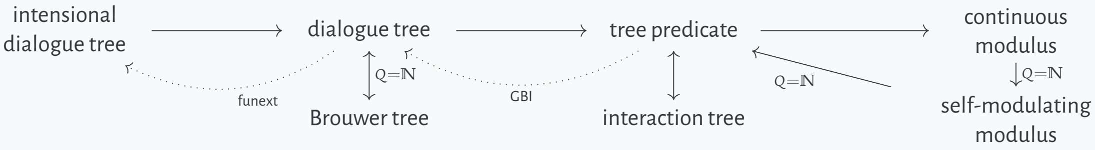
All functions are continous.

Right?

It’s a zoo


intensional dialogue tree
dialogue tree
tree predicate
continuous modulus
funext
Q=ℕ
GBI
Q=ℕ
Q=ℕ
Brouwer tree
interaction tree
self-modulating modulus

Modulus continuity

A function $F: (Q \to A) \to R$

- has modulus $L: \mathbb{L}Q$ at $f: Q \to A$ if for g with $\forall x \in L.\ fx = gx$, we have $Ff = Fg$.
- is *modulus-continuous* if for any f a modulus exists,
- and has a *modulus of continuity function* $M: (Q \to A) \to \mathbb{L}Q$ if for all f, Mf is a modulus at f.

Troelstra & van Dalen (1988)

Sequential continuity

We take any function $\tau : \mathbb{L}A \to (Q + R)$ to describe a tree.

$$\hat{\tau} : (Q \to A) \to \mathbb{N} \to Q + R$$

$$\hat{\tau}\; f\; (\mathsf{S} n) := \hat{\tau}\; (\lambda l.\ \tau(l \mathbin{++} [fq]))\; n \text{ if } \tau\; [\,] = \mathsf{inl}\; q$$

$$\hat{\tau}\; f\; n := \tau\; [\,] \text{ otherwise}$$

F is *sequentially continuous* if $\exists \tau : \mathbb{L}A \to Q + R. \forall f.\ \exists n.\ \hat{\tau}\; f\; n = \mathsf{inr}(Ff)$.

van Oosten (2011)

Dialogue trees

$$\frac{r : R}{\eta r : \mathbb{D}} \qquad \frac{q : Q \qquad k : A \to \mathbb{D}}{\beta q k : \mathbb{D}}$$

$$\partial f\,(\eta o) := o \qquad \partial f\,(\beta q k) := \partial f\,(k(fq))$$

A function F is *eloquent* if there exists $d : \mathbb{D}$ such that $\forall f : Q \to A.\ Ff = \partial f d$.

Escardo (2013), Ghani, Hancock, and Pattinson (2009)

Intensional dialogue trees

$$\frac{r : R}{\eta_{\mathsf{i}} r : \mathbb{D}_{\mathsf{i}}(\lambda f.\ r)}$$

$$\frac{q : Q \qquad k : A \to (Q \to A) \to R \qquad H : \forall a : A.\ \mathbb{D}_{\mathsf{i}}(ka)}{\beta_{\mathsf{i}} qkH : \mathbb{D}_{\mathsf{i}}(\lambda f.\ k(fq)f)}$$

Proof

Theorem

Eloquent functions have a modulus of continuity function.

Proof

Fix $F\colon (Q \to A) \to R$ and let d be such that $\forall f\colon Q \to A.\ Ff = \partial f d$.
We define

$$\partial' f\,(\eta o) := \square \qquad\qquad \partial' f\,(\beta q k) := q :: \partial' f\,(k(fq))$$

Now $Mf := \partial'\ f\ \ d$ is a modulus of continuity function. $\square$

Conjecture: The converse is not provable.

Constructive Reverse Maths propaganda

Theorem

We show that principle P *does not hold in system* I *by providing a model* M *for* I *with* $\neg(M \vDash P)$.

Theorem (Often easier to prove)

We show that principle P *does not hold in system* I *by proving* $I \vdash P \to T$ *for* T *being unprovable in* I.

Theorem (Constructive Reverse Maths)

$I \vdash P \leftrightarrow T$ *for* T *being unprovable in* I.

Δ_1^0-BI

Let $\tau : \mathbb{L}\mathbb{N} \to \mathbb{B}$, closed under extension, i.e. $\tau l \to \tau(l +\!\!+ l')$.

τ is barred if $\forall f : \mathbb{N} \to \mathbb{N}.\ \exists n.\ \tau[f 0, \ldots, f n]$.

τ is inductively barred if $B\ \tau\ []$ with

$$\frac{\tau u}{B\ \tau\ u} \qquad\qquad \frac{\forall a. B\ \tau\ (u +\!\!+ [a])}{B\ \tau\ u}$$

Theorem

Δ_1^0-BI *is equivalent to the statement that every function that has a continuous modulus of continuity function is eloquent.*

Theorem (Brede Herbelin 2021)

BI *is equivalent to* $\mathrm{GBI}(\mathbb{N})$.

Δ°_1-choice

$$\forall R\colon \mathbb{N}\to\mathbb{N}\to\mathbb{B}.\ (\forall x.\exists y.\ Rxy) \to \exists f.\ \forall x.\ Rx(fx)$$

Holds in all foundations (for different reasons).

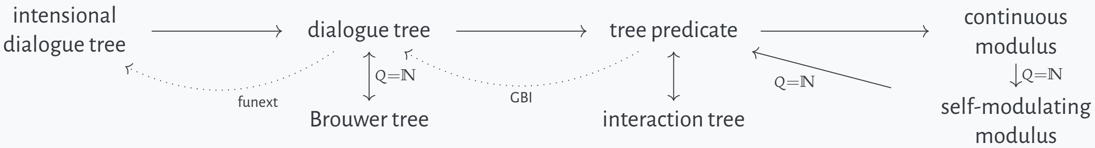
Theorem (Fujiwara Kawai 2019)

A function $F\colon(\mathbb{N}\to\mathbb{N})\to\mathbb{N}$ has a continuous modulus of continuity function if and only if it has a self-modulating modulus of continuity function, uses Δ°_1-choice on $\mathbb{N}$.

Corollary

The following are equivalent:

- *F has a continuous modulus of continuity function*
- *F has a self-modulating modulus of continuity function*
- *F is induced by a computable tree predicate of type $\mathbb{L}\mathbb{N} \to \mathbb{N} + \mathbb{N}$.*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Continuity principles

Theorem

The statement that every function has a continuous modulus of continuity function implies the statement that every function is eloquent (has an underlying inductive dialogue) if and only if

?

Impossibility results

Xu Escardo: There is no M: $(\mathbb{N}\to\mathbb{N})\to\mathbb{N}\to(\mathbb{N}\to\mathbb{N})\to\mathbb{L}\mathbb{N}$
such that for all F: $(\mathbb{N}\to\mathbb{N})\to\mathbb{N}$,
MF is modulus function for F.

Baillon (for type theory): If every function $(P \to \bot) \to \bot$ is modulus continuous, then P.

Brede Herbelin: There are non-continuous functions of type $(\mathbb{N} \to \mathbb{P}) \to \mathbb{N}$.

What types do the functions have?

What do we mean by continuous?

All functions are continous.

Right?

What's an existential quantifier?

Do we believe in classical logic?

Do we believe in bar induction?

CIC propaganda

CIC

- has notions of $\exists$ and Σ that differ
- proves virtually no choice axioms apart from Δ_1^0-choice
- not even countable choice
- not even unique choice
- thus one can make logic classical and keep functions computable

Proof Assistant propaganda

When investigating fine-grained differences, a proof assistant works as a proof *assistant*.

Conclusion

- C: Put results in common foundation, mechanised, generalised types from $\mathbb{N}$
- A: We have at least 3 natural, different notions of continuity.
- Q: Does the Δ_1^0-BI proof lift in general to Δ_1^0-GBI?
- Q: What separates the continuity principles?
- A: Constructive reverse maths Qs are fun, especially in CIC, especially in a proof assistant